American University of Beirut
Department of Computer Science
CMPS 211 - Spring 2015-16
Assignment 3 - Solution Key

Exercise 1 (10 Points)

Let A, B, and C be sets. Show that:

(a) $(A \cup B) \subseteq (A \cup B \cup C)$
Solution:
$\{x|x \in (A \cup B)\}$ (Assumption)
$\rightarrow \{x|x \in A \lor x \in B\}$ (Definition of Union)
$\rightarrow \{x|x \in A \lor x \in B \lor x \in C\}$ (Addition)
$\rightarrow \{x|x \in (A \cup B \cup C)\}$ (Definition of Union)

(b) $(A \cap B \cap C) \subseteq (A \cap B)$
Solution:
$\{x|x \in (A \cap B \cap C)\}$ (Assumption)
$\rightarrow \{x|x \in A \land x \in B \land x \in C\}$ (Definition of Intersection)
$\rightarrow \{x|x \in A \land x \in B\}$ (Simplification)
$\rightarrow \{x|x \in A \cap B\}$ (Definition of Intersection)

(c) $(A - B) - C \subseteq A - C$
Solution:
$\{x|x \in ((A - B) - C)\}$ (Assumption)
$\rightarrow \{x|((x \in A \land x \notin B) \land x \notin C)\}$ (Definition of Difference)
$\rightarrow \{x|((x \in A \land x \notin C)\}$ (Simplification - After use of commut. and Assoc. Laws on $\land$)
$\rightarrow \{x|x \in (A - C)\}$ (Definition of Difference)

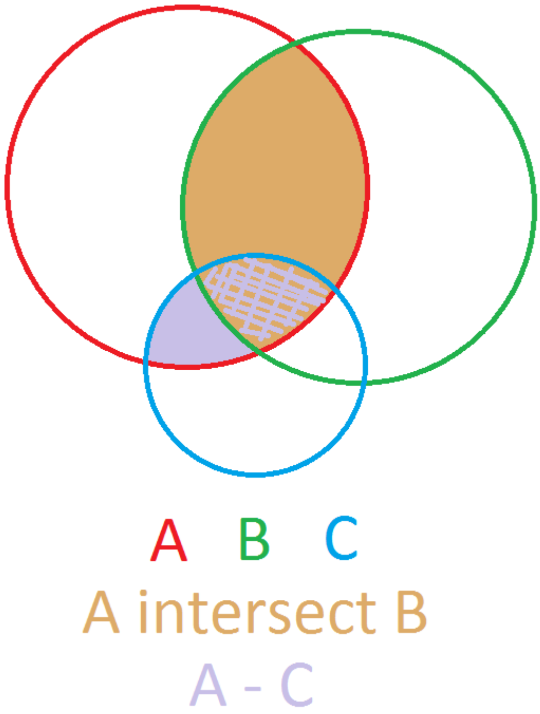
(d) $(A - C) \cap (C - B) = \phi$ *Solution:*
$\{x|x \in (A - C) \cap (C - B)\}$ (Assumption)
$\rightarrow \{x|((x \in A \land x \notin C) \land (x \in C \land x \notin B)\}$ (Definition of Difference and Intersection)
$\rightarrow \{x|(x \in A \land (x \notin C \land x \in C) \land x \notin B\}$ (Commutativity)
$\rightarrow \{x|x \notin C \land x \in C\}$ (Simplification)
Contradiction, Therefore the logical expression is unsatisfiable (always has truth value "false"), which means that $(A - C) \cap (C - B)$ has no elements.
Hence, $(A - C) \cap (C - B) = \phi$

(e) $(B - A) \cup (C - A) = (B \cup C) - A$
$\{x|x \in (B - A) \cup (C - A)\}$ (Assumption)
$\leftrightarrow \{x|(x \in B \land x \notin A) \lor (x \in C \land x \notin A)\}$ (Definition of Difference and Union)
$\leftrightarrow \{x|(x \in B \lor x \in C) \land x \notin A\}$ (Distribution)
$\leftrightarrow \{x|x \in (B \cup C) - A\}$ (Definition of Difference and Union)
Since every step is an equivalence then $(B - A) \cup (C - A) = (B \cup C) - A$

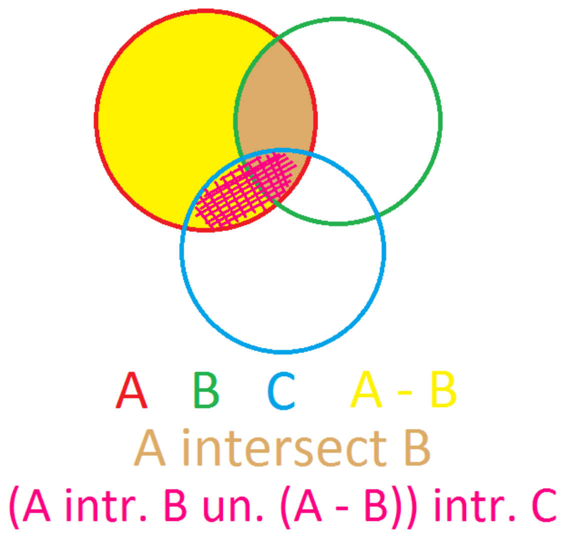
Exercise 2 (10 Points)

Draw The Venn Diagrams for each of these combinations of the sets A, B, and C.

(a) $(A \cap B) \cup (A - C)$
{the solution is the shaded areas}



(b) $(A \cap B \cup (A - B)) \cap C$
{the solution is the shaded areas}



Note that $A \cap B \cup (A - B)$ is equal to A.

Exercise 3 (15 Points)

Give an example of two uncountable sets A and B such that $A - B$ is:

(a) Finite $\mathbf{A} = \mathbb{R}, \mathbf{B} = \mathbb{R} - \{\mathbf{0}\}$. $\mathbf{A} - \mathbf{B} = \{\mathbf{0}\}$(**Finite**)

(b) Countable infinite $\mathbf{A} = \mathbb{R}, \mathbf{B} = \mathbb{R} - \mathbb{N}$. $\mathbf{A} - \mathbf{B} = \mathbb{N}$(**CountablyInfinite**)

(c) Uncountable infinite $\mathbf{A} = \mathbb{R}, \mathbf{B} = \mathbb{R} - [\mathbf{0}, \mathbf{1}]$. $\mathbf{A} - \mathbf{B} = [\mathbf{0}, \mathbf{1}]$(**Uncountable**)

Exercise 4 (15 Points)

Prove that if A is uncountable and $A \subseteq B$, then B is uncountable.

Solution 1: Proof by contradiction
Assume A is uncountable and $A \subseteq B$. For the sake of contradiction, assume B is countable. if B is countable then there exists a bijection (one-to-one and onto) (f) between B and $\mathbb{N}$.

$$f : B \to \mathbb{N}$$

Construct the function $f|_A$ by restricting the domain of f to $A \subseteq B$, i.e:

$$f|_A : A \to \mathbb{N} : f|_A(x) = f(x)$$

$f|_A$ is defined the same was as f but with a smaller domain. Since f is injective (one-to-one), then $f|_A$ is also one-to-one (note that $f|_A$ may no longer be onto).
However, since A is uncountable, then there does not exist any injective function from it to $\mathbb{N}$. Contradiction, B must be uncountable.

Solution 2:
Assume that B is countable. Then the elements of B can be listed as b_1, b_2, b_3 Because A is a subset of B, taking the subsequence of $\{b_n\}$ that contains the terms that are in A only gives a listing of the elements of A. Because A is uncountable, this is impossible, and therefore B can't be countable

Exercise 5 (15 Points)

Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

(a) The negative integers. **Countably infinite:** $\mathbf{f(x) = -x}$

(b) The even integers. **Countably infinite:** $\mathbf{f(x)} = \begin{cases} x & \text{if } \mathbf{x \text{ is even}} \\ -x - 1 & \text{if } \mathbf{x \text{ is odd}} \end{cases}$

(c) The real numbers between 0 and 1. **Uncountable**

(d) The positive integers less than 1,000,000,000. **Countable, because they are finite**

(e) The integers that are multiples of 7. **Countably infinite:** $\mathbf{f(x)} = \begin{cases} 7 \times \frac{x}{2} & \text{if } \mathbf{x \text{ is even}} \\ 7 \times \frac{-x-1}{2} & \text{if } \mathbf{x \text{ is odd}} \end{cases}$

(f) The odd negative integers. **Countably infinite:** $\mathbf{f(x) = -2x + 1}$

(g) The integers with absolute value less than 1,000,000. **Countable, because they are finite**

(h) The set $A \times Z^+$ where $A = 2, 3$. **Countably infinite:** $\mathbf{f(x)} = \begin{cases} (2, \frac{x}{2}) & \text{if } \mathbf{x \text{ is even}} \\ (3, \frac{x+1}{2}) & \text{if } \mathbf{x \text{ is odd}} \end{cases}$

(i) All bit strings not containing the bit 0. **Countable:** for any integer n, the corresponding bit string would be a string formed of n $0s$

Exercise 6 (15 Points)

Prove that the set of all polynomials of degree ≤ 2 with integer coefficients is countable. [Hint: use the proof that Q is countable shown in the slides]

Solution:
Let P_2 be the set of all polynomials of degree ≤ 2 with integer coefficients.
Thus $P_2 = \{ax^2 + bx + c \mid a, b, c \text{ are integers}\}$.
Since the set $\mathbb{N} \times \mathbb{N}$ is countable, therefore $\mathbb{Z} \times \mathbb{Z}$ is countable also countable ($|\mathbb{N}| = |\mathbb{Z}|$ since there is a bijection between the two sets, and therefore we can substitute $\mathbb{N}$ by $\mathbb{Z}$). Again, $\mathbb{Z}^2$ is countable, $\mathbb{Z}^2 \times \mathbb{Z} = \mathbb{Z}^3$ is also countable.

Define a function f: $\mathbb{Z}^3 \rightarrow P_2$ by the formula:

$$F(a, b, c) = ax^2 + bx + c$$

where $a, b, c \in \mathbb{Z}$
By construction the function f is a one-to-one correspondence. Hence P_2 is countable.

Exercise 7 (10 Points)

Can you conclude that A = B if A, B, and C are sets such that:

(a) $A \cup C = B \cup C$. *Solution:*
No, Counter example: $A = \{1, 2\}, C = \{1, 2, 3\}, B = \{3\}$. $A \cup C = B \cup C = \{1, 2, 3\}$

(b) $A \cap C = B \cap C$. *Solution:*
No, Counter example: $A = \{1, 2, 3\}, C = \{1, 2\}, B = \{1, 2, 4\}$. $A \cap C = B \cap C = \{1, 2\}$

(c) $A - C = B - C$. *Solution:*
No, Counter example: $A = \{1, 2, 3\}, C = \{3, 4\}, B = \{1, 2, 4\}$. $A - C = B - C = \{1, 2\}$

Exercise 8 (10 Points)

Find the domain and range of these functions.

(a) The function that assigns to each non-negative integer its first digit.
Domain is the set of non-negative integers (naturals). Range = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (Note that non-negative integers contain the integer 0, and the first digit of 0 is 0, so 0 is in the range).

(b) The function that assigns to a bit string the number of one bits in the string.
Domain is all bit strings. Range is the naturals.

(c) The function that assigns to each non-negative integer its 2nd power and returns the last digit
Domain is the set of non-negative integers (naturals). Range = $\{0, 1, 4, 9, 6, 5\}$. You can find the set by taking all possible last digits (0 to 9), getting the square of the digit, and finding its unit digit.
Recall the expansion of $(10a + b)^2$, *where* a *and* b *are integers, and* $0 \leq b < 10$

(d) The function that raises 2 to the non-negative integer assigned to the function
Domain is the set of non-negative integers (naturals). Range is the powers of 2 (1, 2, 4, 8 ...).

Exercise 9 (10 Points)

Give an example of a function from $\mathbb{Z}$ to $\mathbb{N}$ that is

(a) One-to-one but not onto.

$$f : \mathbb{Z} \to \mathbb{N} : f(x) = \begin{cases} 2x + 2 & \text{if } x \geq 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$$

0 has no preimage

(b) Onto but not One-to-One.

$$f : \mathbb{Z} \to \mathbb{N} : f(x) = |x|$$

$$\text{for any} x,\ f(x) = f(-x)$$

(c) Both onto and one-to-one (but different from the identity function).

$$f : \mathbb{Z} \to \mathbb{N} : f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$$

(d) Neither one-to-one nor onto.

$$f : \mathbb{Z} \to \mathbb{N} : f(x) = 0$$

Exercise 10 (10 Points)

Let $f(x) = ax^2 + bx + c$ and $g(x) = 3dx - 2e$, where a, b, c, d and e are constants. Determine necessary and sufficient conditions on the constants a, b, c, d and e so that $f \circ g = g \circ f$.

Solution:
$f \circ g = a(3dx - 2e)^2 + b(3dx - 2e) + c = 9ad^2x^2 + (3bd - 12ade)x + (4ae^2 - 2be + c)$
$g \circ f = 3d(ax^2 + bx + c) - 2e = 3adx^2 + 3bdx + (3cd - 2e)$

Therefore, if $f \circ g = g \circ f$, then the following should be equal:

- $9ad^2 = 3ad \rightarrow (3d = 1 \lor a = 0 \lor d = 0)$
- $3bd - 12ade = 3bd \rightarrow (12ade = 0) \leftrightarrow (a = 0 \lor d = 0 \lor e = 0)$
- $(4ae^2 - 2be + c) = (3cd - 2e) \rightarrow 2e(2ae - b + 1) = c(3d - 1)$

Exercise 11 (10 Points)

Determine whether the symmetric difference is associative; that is, if A, B, and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$? Note: The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

Solution:

A	B	C	$A \oplus B$	$B \oplus C$	$A \oplus (B \oplus C)$	$(A \oplus B) \oplus C$
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	0	0	0
1	0	0	0	1	1	1
1	0	1	1	1	0	0
1	1	0	0	1	0	0
1	1	1	0	0	1	1

Exercise 12 (10 Points)

Write a closed form notation for the following summations

(a)

$$\sum_{i=1}^{2n}(1 - 1^i) = \sum_{i=1}^{2n}(1 - 1) = \sum_{i=1}^{2n} 0 = 0$$

(b)

$$\sum_{i=1}^{n}(3^i - 2^i) = \sum_{i=1}^{n} 3^i - \sum_{i=1}^{n} 2^i = 3 \cdot \frac{3^n - 1}{2} - 2 \cdot (2^n - 1)$$

(c)

$$\sum_{i=1}^{n}\sum_{j=1}^{m} ij = \sum_{i=1}^{n} i \sum_{j=1}^{m} j = \frac{n(n+1)}{2}\frac{m(m+1)}{2} = \frac{n(n+1)m(m+1)}{4}$$

(d)

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(2i+3j) = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} 2i + \sum_{j=1}^{m} 3j\right)$$

$$= \sum_{i=1}^{n}\left(2i\sum_{j=1}^{m} 1 \;+\; 3\sum_{j=1}^{m} j\right)$$

$$= \sum_{i=1}^{n}\left(2im \;+\; \frac{3m(m+1)}{2}\right)$$

$$= \sum_{i=1}^{n}(2im) \;+\; \sum_{i=1}^{n}\left(\frac{3m(m+1)}{2}\right)$$

$$= 2m\sum_{i=1}^{n} i \;+\; \left(\frac{3m(m+1)}{2}\right)\sum_{i=1}^{n} 1$$

$$= 2m\frac{n(n+1)}{2} \;+\; \frac{3m(m+1)n}{2}$$

$$= \frac{2mn(n+1)}{2} \;+\; \frac{3m(m+1)n}{2}$$

$$= \frac{mn(n+1) \;+\; 3m(m+1)n}{2}$$

(e) Compact solution - Can be expanded step by step as in part (d)

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(2i^3+3j^2) = m\sum_{i=1}^{n} 2i^3 + n\sum_{j=1}^{m} 3j^2$$

$$= 2m\sum_{i=1}^{n} i^3 + 3n\sum_{j=1}^{m} j^2$$

$$= 2m(\frac{n^2(n+1)^2}{4}) + 3n(\frac{m(m+1)(2m+1)}{6})$$

Exercise 13 **(10 Points)**

Write the product notation of $C(n,r) = \frac{n!}{r! \times (n-r)!}$

Solution:

$$C(n,r) = \frac{n!}{r! \times (n-r)!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-r+1)}{r!}$$

$$\frac{\prod_{i=1}^{n} i}{\prod_{j=1}^{r} j \times \prod_{k=1}^{n-r} k} = \frac{\prod_{i=n-r+1}^{n} i}{\prod_{j=1}^{r} j} = \prod_{i=1}^{r} \frac{n-i+1}{i}$$

Exercise 14 **(10 Points)**

Prove or disprove each of these statements about the floor and ceiling functions.

(a) $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$ for all real numbers x.

We have $\lceil x \rceil \in \mathbb{Z}$ By definition. Also, $\lfloor z \rfloor = z$ for any $z \in \mathbb{Z}$
Therefore $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$ for all reals.

(b) $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ for all real numbers x and y.

Take $x = 1.5$, $y = 1.8$. $\lfloor x + y \rfloor = 3$. $\lfloor x \rfloor + \lfloor y \rfloor = 2$. Disproved by counter-example.

(c) $\left\lceil \frac{\lceil \frac{x}{2} \rceil}{2} \right\rceil = \lceil \frac{x}{4} \rceil$ for all real numbers x.

Every real number can be written as $4q + r$ where $q \in \mathbb{Z}$ and $0 \leq r < 4$ ($r \in \mathbb{R}$).

$$\left\lceil \frac{\lceil \frac{4q+r}{2} \rceil}{2} \right\rceil = \left\lceil \frac{\lceil 2q + \frac{r}{2} \rceil}{2} \right\rceil = \left\lceil \frac{2q + \lceil \frac{r}{2} \rceil}{2} \right\rceil = \left\lceil q + \frac{\lceil \frac{r}{2} \rceil}{2} \right\rceil = q + \left\lceil \frac{\lceil \frac{r}{2} \rceil}{2} \right\rceil$$

$$\textit{since } q \in \mathbb{Z}$$

Also:

$$\left\lceil \frac{4q+r}{4} \right\rceil = \left\lceil q + \frac{r}{4} \right\rceil = q + \left\lceil \frac{r}{4} \right\rceil$$

Therefore, the property holds for all real numbers if and only if $\left\lceil \frac{\lceil \frac{r}{2} \rceil}{2} \right\rceil = \lceil \frac{r}{4} \rceil$ holds for the real numbers in $[0, 4)$.
Take the case where $x = 0$. Both sides will be 0, so it holds.
Take the case $0 < x \leq 2$:

$$\left\lceil \frac{\lceil \frac{x}{2} \rceil}{2} \right\rceil = \left\lceil \frac{1}{2} \right\rceil = 1 \text{ and } \left\lceil \frac{x}{4} \right\rceil = 1$$

Finally, take the case where $2 < x < 4$:

$$\left\lceil \frac{\lceil \frac{x}{2} \rceil}{2} \right\rceil = \left\lceil \frac{2}{2} \right\rceil = 1 \text{ and } \left\lceil \frac{x}{4} \right\rceil = 1$$

Therefore, it holds for all real numbers in $[0, 4)$ (Exhaustive Proof), implying that the original statement holds for all real numbers.

(d) $\lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{x} \rfloor$ for all real numbers x.

Take $x = 3.1$. $\lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{\lceil 3.1 \rceil} \rfloor = \lfloor \sqrt{4} \rfloor = 2$.
Note that $1.8^2 = 3.24 > 3.1$. Therefore $\sqrt{3.1} < 1.8$, $\lfloor \sqrt{1.8} \rfloor \leq 1 < 2$.
Disproved by a Counter example.

(e) $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$ for all real numbers x and y.
Take $x = a + \alpha$, and $y = b + \beta$, where $a, b \in \mathbb{Z}$ and $0 \leq \alpha, \beta < 1$

$$\begin{gathered} \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor \\ \leftrightarrow \lfloor a + \alpha \rfloor + \lfloor b + \beta \rfloor + \lfloor a + \alpha + b + \beta \rfloor \leq \lfloor 2a + 2\alpha \rfloor + \lfloor 2b + 2\beta \rfloor \\ \leftrightarrow a + \lfloor \alpha \rfloor + \lfloor \beta \rfloor + a + b + \lfloor \alpha + \beta \rfloor \leq 2a + \lfloor 2\alpha \rfloor + 2b + \lfloor 2\beta \rfloor \quad \textit{since } a, b \in \mathbb{Z} \\ \leftrightarrow \lfloor \alpha \rfloor + \lfloor \beta \rfloor + \lfloor \alpha + \beta \rfloor \leq + \lfloor 2\alpha \rfloor + \lfloor 2\beta \rfloor \end{gathered}$$

Now we can use proof by cases (6 cases, but can be reduced to 4 without loss of generality):

1. $(0 \leq \alpha < 0.5) \ \wedge \ (0 \leq \beta < 0.5) \rightarrow$
   - $\lfloor \alpha \rfloor = \lfloor \beta \rfloor = 0$
   - $0 \leq \alpha + \beta < 1 \rightarrow \lfloor \alpha + \beta \rfloor = 0$
   - $0 \leq 2\alpha, 2\beta < 1 \rightarrow \lfloor 2\alpha \rfloor = \lfloor 2\beta \rfloor = 0$

   And therefore $0 \leq 0$ which is true

2. $(0.5 \leq \alpha < 1) \ \wedge \ (0.5 \leq \beta < 1) \rightarrow$
   - $\lfloor \alpha \rfloor = \lfloor \beta \rfloor = 0$
   - $1 \leq \alpha + \beta < 2 \rightarrow \lfloor \alpha + \beta \rfloor = 1$
   - $1 \leq 2\alpha, 2\beta < 2 \rightarrow \lfloor 2\alpha \rfloor = \lfloor 2\beta \rfloor = 1$

   And therefore $0 + 1 \leq 1 + 1 \rightarrow 1 \leq 2$ which is true

3. w.l.o.g, $(0 \leq \alpha < 0.5) \ \wedge \ (0.5 \leq \beta < 1) \ \wedge \ (\alpha + \beta \geq 1) \rightarrow$
   - $\lfloor \alpha \rfloor = \lfloor \beta \rfloor = 0$
   - $1 \leq \alpha + \beta < 2 \rightarrow \lfloor \alpha + \beta \rfloor = 1$
   - $0 \leq 2\alpha < 1 \rightarrow \lfloor 2\alpha \rfloor = 0$
   - $1 \leq 2\beta < 2 \rightarrow \lfloor 2\beta \rfloor = 1$

   And therefore $0 + 1 \leq 0 + 1 \rightarrow 1 \leq 1$ which is true

4. w.l.o.g, $(0 \leq \alpha < 0.5) \ \wedge \ (0.5 \leq \beta < 1) \ \wedge \ (\alpha + \beta < 1) \rightarrow$
    - $\lfloor \alpha \rfloor = \lfloor \beta \rfloor = 0$
    - $0 \leq \alpha + \beta < 1 \rightarrow \lfloor \alpha + \beta \rfloor = 0$
    - $0 \leq 2\alpha < 1 \rightarrow \lfloor 2\alpha \rfloor = 0$
    - $1 \leq 2\beta < 2 \rightarrow \lfloor 2\beta \rfloor = 1$

    And therefore $0 + 0 \leq 0 + 1 \rightarrow 0 \leq 1$ which is true

Therefore, it has been proven by cases that $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$ for all real numbers x and y